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# APPLICATIONS OF MAÑÉ'S THEOREM

YESHUN SUN AND YONGCHENG YIN

**ABSTRACT.** In this paper, we show that the Julia set of a critically non-recurrent rational map on Riemann sphere is either shallow with respect to spherical metric or itself the whole sphere. Rigidity and ergodicity results are also obtained for such maps.

**Key words and phrases.** Julia set, critically non-recurrent rational map, distortion, shallow, rigidity, ergodicity.

## 1. INTRODUCTION

Let  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  be a rational map with degree  $\deg(f) \geq 2$ . A point  $z$  is said to be a *periodic point* if  $f^k(z) = z$  for some  $k \geq 1$ . The minimal of such  $k$  is called the *period* of  $z$ . For a periodic point  $z_0$ , denote the *multiplier* of  $z_0$  by  $\lambda = (f^k)'(z_0)$ . The periodic point  $z_0$  is either *attracting*, *indifferent* or *repelling* according to  $|\lambda| < 1$ ,  $|\lambda| = 1$  or  $|\lambda| > 1$ . In the indifferent case, we say  $z_0$  is *parabolic* if  $\lambda$  is the root of unity.

The *Julia set*, denoted by  $J(f)$ , is the closure of set of repelling periodic points. Its complement is called *Fatou set*, denoted by  $F(f)$ . A connected component of  $F(f)$  is called a *Fatou component*. D. Sullivan proved that each Fatou component  $U$  is preperiodic, i.e., there exist integers  $m \geq 1, n \geq 0$  so that  $f^{m+n}(U) = f^n(U)$  and every periodic Fatou component is either an attracting basin, a parabolic basin, a Siegel disk or a Herman ring. For the classical results of complex dynamics, see [B], [Mi] and [CG].

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YESHUN SUN AND YONGCHENG YIN

The *post-critical set*  $P(f)$  of  $f$ , i.e., the closure of the forward orbits of critical points, will play the crucial rule in the study of complex dynamics. Recall that the  $\omega$ -limit set  $\omega(c)$  of  $c$  is  $\{z \in \overline{\mathbb{C}} \mid \text{there exists } n_k \rightarrow \infty \text{ such that } z = \lim_{k \rightarrow \infty} f^{n_k}(c)\}$ .

**Definition 1.1.** A rational map  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  is called *critically non-recurrent* if each critical point  $c \in J(f)$  is non-recurrent, i.e.,  $c \notin \omega(c)$ .

We equip Riemann sphere  $\overline{\mathbb{C}}$  with the standard spherical metric  $ds = \frac{2}{1+|z|^2}|dz|$ . If we identify the unit sphere  $S^2$  in  $\mathbb{R}^3$  with the Riemann sphere, then this metric coincides with the spherical distance and is invariant under  $SO(3)$  as well as  $1/z$ . Restricted to any proper compact subset of Riemann sphere, this metric is equivalent to the Euclidean metric. Denote  $B(x, \varepsilon)$  the ball centered at  $x$  and of radius  $\varepsilon$  in the spherical metric on  $\overline{\mathbb{C}}$  and  $D(x, \varepsilon)$  the disk centered at  $x$  and of radius  $\varepsilon$  in the Euclidean metric on  $\mathbb{C}$ .

The following definition is given by C. McMullen[Mc].

**Definition 1.2.** A compact subset  $X \subset \overline{\mathbb{C}}$  is *shallow* if there exists  $0 < k < 1$  such that any ball  $B(z, r)$ , where  $z \in X$  and  $0 < r < 1$ , contains a smaller ball  $B(y, kr) \subset \overline{\mathbb{C}} \setminus X$ .

It's easy to prove that if  $X$  is shallow, then its box dimension  $\text{BD}(X) < 2$  and its Hausdorff dimension  $\text{HD}(X) \leq \text{BD}(X) < 2$ .

Our first result on critically non-recurrent rational maps is the following.

**Theorem 1.1.** *For a critically non-recurrent rational map  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ , the Julia set  $J(f)$  is either shallow or  $J(f) = \overline{\mathbb{C}}$ .*

An immediate consequence of Theorem 1.1 is:

**Corollary 1.1** (Cf. [U]). *For a critically non-recurrent rational map  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ ,  $\text{HD}(J(f)) < 2$  or  $J(f) = \overline{\mathbb{C}}$ .*

## APPLICATIONS OF MAÑÉ'S THEOREM

*Remark 1.1.* C. McMullen [Mc] and F. Przytycki and S. Rohde [PR] proved the same theorems for quadratic polynomials with Siegel disk of bounded type rotation numbers and Collet-Eckmann rational maps without parabolic points.

Quasiconformality is a relaxation of conformality in complex analysis. It turns out to be very useful in the study of complex dynamics and can be defined in more general setting.

**Definition 1.3.** A homeomorphism  $\phi : X \rightarrow X$  of metric space  $(X, d)$  is called *quasiconformal* if there exists a constant  $K$  such that

$$\limsup_{r \rightarrow 0} \frac{\max_{d(y,x)=r} d(\phi(y), \phi(x))}{\min_{d(y,x)=r} d(\phi(y), \phi(x))} \leq K$$

for all  $x \in X$ .

A conjugacy quasiconformal off  $J(f)$  being actually globally quasiconformal reflects the rigidity of the systems. The next result is about the rigidity of critically non-recurrent rational maps.

**Theorem 1.2.** *Let  $f$  be a critically non-recurrent rational map. If the orientation preserving homeomorphism  $\phi : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  conjugates  $f$  to a rational map  $g$ , then  $\phi$  is isotopic relative to  $J(f)$  to a quasiconformal conjugacy.*

*Remark 1.2.* P. Haïssinsky [H] obtained the same result for uniformly weakly hyperbolic rational maps defined therein in different way. F. Przytycki and S. Rohde [PR1] proved it for topological Collet-Eckmann rational maps.

Recall that a homeomorphism  $\phi : X \rightarrow X$  is *isotopic* to  $\psi : X \rightarrow X$  relative to a subset  $A \subset X$  if there exists a continuous map  $H : X \times [0, 1] \rightarrow X$  such that  $H(x, 0) = \phi$  and  $H(x, 1) = \psi$  for all  $x \in X$  and  $H(x, t) = x$  for all  $x \in A$  and  $0 \leq t \leq 1$ .

The Lebesgue measure on Riemann sphere induced by the spherical metric is denoted by  $\text{mes}(X)$  for measurable subset  $X \subset \overline{\mathbb{C}}$ . Therefore the full measure of the Riemann sphere is  $4\pi$ .

**Definition 1.4.** A rational map  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  is *ergodic* if any measurable set  $A$  satisfying  $f^{-1}(A) = A$  has zero or full measure in the sphere.

Now we come to our ergodicity result on critically non-recurrent rational maps.

**Theorem 1.3.** *For a critically non-recurrent rational map  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ , either the Julia set  $J(f)$  is the whole sphere and  $f$  is ergodic or the Hausdorff dimension of the Julia set  $J(f)$  is strictly less than 2.*

The following notations will be used in the subsequent sections.

- (1) Reserve  $d(d_E, \text{resp.})$  to denote the spherical(Euclidean, resp.) distance on  $\overline{\mathbb{C}}(\mathbb{C}, \text{resp.})$  and  $\text{diam } X(\text{diam}_E X, \text{resp.})$  the corresponding diameter of the subset  $X \subset \overline{\mathbb{C}}(X \subset \mathbb{C}, \text{resp.})$ .
- (2) The distortion  $\text{Dist}(X, z)(\text{Dist}_E(X, z), \text{resp.})$  of a compact subset  $X \subset \overline{\mathbb{C}}(X \subset \mathbb{C}, \text{resp.})$  about  $z \notin X$  is

$$\frac{\max_{y \in X} d(y, z)}{\min_{y \in X} d(y, z)} \left( \frac{\max_{y \in X} |y - z|}{\min_{y \in X} |y - z|}, \text{resp.} \right).$$

- (3) Two positive numbers  $A$  and  $B$  are  $K$ -commensurable or simply commensurable if  $K^{-1} \leq A/B \leq K$  for some  $K \geq 1$  independent of  $A$  and  $B$ .  $A \asymp B$  means  $A$  and  $B$  are commensurable.

## 2. BACKGROUND MATERIALS

**2.1. Distortion lemmas.** The distortion theorem of univalent maps is a powerful tool in the study of complex dynamics. The following distortion theorem of the version for  $p$ -valent maps is well-known(See [CJY] or Appendix C in [ST]).

**Lemma 2.1.** *Let  $U_1$  and  $U_2$  be simply connected domains in  $\overline{\mathbb{C}}$  and  $g : U_1 \rightarrow U_2$  be a proper holomorphic map of  $\deg g \leq p$ . Then*

$$\{w | \rho_{U_2}(w, g(z_0)) \leq C^{-1}\} \subset g(\{z | \rho_{U_1}(z, z_0) \leq r\}) \subset \{w | \rho_{U_2}(w, g(z_0)) \leq 1\}$$

*for any  $z_0 \in U_1$ , where  $\rho_{U_1}$  and  $\rho_{U_2}$  are hyperbolic metrics of  $U_1$  and  $U_2$ , and  $C$  is a constant depending only on  $p$  and  $r$ .*

## APPLICATIONS OF MAÑÉ'S THEOREM

Denote  $\rho$  the hyperbolic metric in  $D(0, 1)$  and  $B_\rho(0, r)$  the hyperbolic ball centered at 0 of radius  $r$ .

**Lemma 2.2.** *Let  $h : D(0, 1) \rightarrow D(0, 1)$  be a proper holomorphic map of degree at most  $p$  and  $h(0) = 0$ . Then there exists a constant  $R^*$  depending only on  $p$  and  $R$  such that  $D(0, R^*) \subset h(B_\rho(0, R))$  and the component of  $h^{-1}(D(0, R^*))$  containing 0 is a subset of  $B_\rho(0, R)$ .*

*Proof.* Assume  $\deg h = p' \leq p$ . In the case  $p' = 1$ , the conclusion follows immediately from Koebe one-quarter theorem of univalent maps. Now we suppose  $p' \geq 2$ . Let  $0 = z_0, z_1, \dots, z_{p'-1}$  be the preimages of  $h(z_0) = 0$  and  $h_j : D(0, 1) \rightarrow D(0, 1)$  be the univalent map so that  $h_j(z_j) = 0, 0 \leq j \leq p' - 1$ . Then

$$h(z) = e^{i\theta} \prod_{j=0}^{p'-1} h_j(z)$$

for some  $\theta \in \mathbb{R}$ .

There are at most  $p'$  points of  $\{z_j | 0 \leq j \leq p' - 1\}$  in the closed hyperbolic ball  $\overline{B_\rho(0, R)}$ . We can find a hyperbolic circle  $\gamma$  centered at 0 in  $B_\rho(0, R)$  such that

$$\rho(z, z_j) \geq \frac{R}{2(p' - 1)}$$

for any  $z \in \gamma$  and  $0 \leq j \leq p' - 1$ . Then for all  $z \in \gamma$ ,

$$\rho(h_j(z), 0) = \rho(h_j(z), h_j(z_j)) = \rho(z, z_j) \geq \frac{R}{2(p' - 1)}.$$

Going back to the Euclidean metric, we have

$$d_E(h_j(z), 0) \geq \frac{e^{\frac{R}{2(p'-1)}} - 1}{e^{\frac{R}{2(p'-1)}} + 1} = C_1.$$

Then

$$d_E(h(z), 0) = \prod_{j=0}^{p'-1} d_E(h_j(z), 0) \geq C_1^{p'} = R^*(p', R).$$

and  $R^* = \min\{R^*(p', R) : 1 \leq p' \leq p\}$  is the constant needed.  $\square$

From Lemma 2.2, we have a useful corollary in spherical metric.

**Corollary 2.1.** *Fix the positive integer  $p$  and  $\varepsilon_0 > 0$ . Let  $U$  be a simply connected subset of  $\overline{\mathbb{C}}$  missing a spherical disk of radius  $\varepsilon_0$  and  $g : U \rightarrow B(w_0, 2\delta)$  be a proper holomorphic map of  $\deg g \leq p$ ,  $w_0 = g(z_0)$ , and  $\delta < \frac{\pi - \varepsilon_0}{2}$ . If  $g$  maps  $B(z_0, r) \subset U$  into  $B(w_0, \delta)$ , then there exist constants  $K$  depending only on  $p$  and  $\varepsilon_0$ , and  $r^* \geq \text{diam } g(B(z_0, r))/K$  so that  $B(w_0, r^*) \subset g(B(z_0, r))$  and the component of  $g^{-1}(B(w_0, r^*))$  containing  $z_0$  is a subset of  $B(z_0, r)$ .*

*Proof.* By rotating  $U$  and  $B(w_0, 2\delta)$  respectively to avoid the spherical disk at infinity and of radius  $\varepsilon_0$ , we prove the statement in Euclidean metric first.

Assume  $(\partial g(D(z_0, r))) \cap (\partial D(w_0, \delta)) \neq \emptyset$  at first. Let  $U'$  be the component of  $g^{-1}(D(w_0, \delta))$  containing  $z_0$ . Then  $\text{mod}(U \setminus U') \geq \frac{\log 2}{2\pi p}$ . Therefore,  $\text{diam}_{\rho_U} D(z_0, r) \asymp 1$  and, from distortion lemmas for univalent maps,  $\rho_U(x, z_0) \asymp 1$  for any point  $x \in \partial D(z_0, r)$ , for a constant  $C(p)$  depending only on  $p$ . Take  $R = \min_{x \in \partial D(z_0, r)} \rho_U(x, z_0)$ . By Lemma 2.2, there exists  $r^* = 2\delta R^*$  such that  $D(w_0, r^*) \subset g(D(z_0, r))$  and the component of  $g^{-1}(D(w_0, r^*))$  containing  $z_0$  is a subset of  $D(z_0, r)$ . Denote  $K = 1/R^*$ . Then  $r^* \geq \text{diam } g(D(z_0, r))/K$ .

In the case  $\partial g(D(z_0, r)) \cap \partial D(w_0, \delta) = \emptyset$ , take  $\delta' = d(w_0, \partial g(D(z_0, r))) \leq \delta$ . We replace  $U$  by the component of  $g^{-1}(D(w_0, 2\delta'))$  containing  $z_0$ . Using the same method as above, we get the constant  $K$ .

Coming back to the spherical metric, we know the constant  $K$  depends also on  $\varepsilon_0$ . □

**2.2. Mañé's theorem.** In [Ma](see also [ST]), R. Mañé proved a beautiful theorem. It is the key lemma in the proof of our main results.

**Lemma 2.3** (Mañé's theorem). *Let  $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  be a rational map of degree at least two and  $p$  an integer depending only on  $f$ . If a point  $x \in J(f)$  is not a parabolic periodic point and is not contained in the  $\omega$ -limit set of a recurrent critical point, then for all  $\varepsilon > 0$  there exists*

## APPLICATIONS OF MAÑÉ'S THEOREM

$\delta > 0$ , such that for each  $n \geq 0$  and each connected component  $U$  of  $f^{-n}(B(x, \delta))$ ,

- (1) the spherical diameter of  $U$  is  $< \varepsilon$  and  $\deg(f^n : U \rightarrow B(x, \delta)) \leq p$ , and
- (2) for all  $\varepsilon_1 > 0$  there exists  $n_0 > 0$  such that if  $n \geq n_0$ , the spherical diameter of component  $U$  of  $f^{-n}(B(x, \delta))$  is  $\leq \varepsilon_1$ .

Note that for a critically non-recurrent rational map, a common  $\delta$  can be taken in above lemma for all points in a compact subset of Julia set containing no parabolic points.

**2.3. Dynamics near parabolic points.** Let  $f_0 : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$  be a rational map with a non-degenerate parabolic fixed point 0, i.e.,  $f'_0(0) = 1$  and  $f''_0(0) \neq 0$ . Choose a neighborhood  $U_0$  of 0 so that  $f_0$  maps  $U_0$  homeomorphically onto a neighborhood  $U'_0$ .

**Lemma 2.4** (Fatou coordinates). *Take  $f_0, U_0$  and  $U'_0$  as above. Then there exist simply connected domains  $D_{\pm}$  compactly contained in  $U_0 \cap U'_0$ , whose union forms a punctured neighborhood of 0, satisfying*

$$f_0^{\pm}(\bar{D}_{\pm}) \subset D_{\pm} \cup \{0\} \text{ and } \bigcap_{n \geq 0} f_0^{\pm n}(\bar{D}_{\pm}) = \{0\}.$$

Moreover, there exist univalent maps  $\Phi_{\pm} : D_{\pm} \rightarrow \mathbb{C}$  such that

- (1)  $\Phi_{\pm}(f_0(z)) = \Phi_{\pm}(z) + 1$ ,
- (2)  $\text{Range}(\Phi_+) \supset \{\zeta \mid -\frac{3}{4}\pi < \arg(\zeta - R_0) < \frac{3}{4}\pi\}$  and  $\text{Range}(\Phi_-) \supset \{\zeta \mid \frac{1}{4}\pi < \arg(\zeta + R_0) < \frac{7}{4}\pi\}$  for some  $R_0 > 0$ .

The domains  $D_+$  and  $D_-$  in Lemma 2.4 are called attracting petal and repelling petal respectively. We will make use of this coordinates to deal with the parabolic points in critically non-recurrent rational maps.

Set  $\tilde{V}_0^* = \{\zeta \mid \text{Re}\zeta < -R_0\}$ ,  $\tilde{V}_0 = \{\zeta \mid \text{Re}\zeta < -R_0, |\text{Im}\zeta| > R_0\}$ ,  $\tilde{\Omega}_0 = \{\zeta \mid -R_0 - 1 < \text{Re}\zeta < -R_0, |\text{Im}\zeta| < R_0\}$ ,  $\tilde{\Omega}_j = \tilde{\Omega}_0 - j, j \geq 1$  and  $V_0^* = \Phi_-^{-1}(\tilde{V}_0^*)$ ,  $V_0 = \Phi_-^{-1}(\tilde{V}_0)$ ,  $\Omega_j = \Phi_-^{-1}(\tilde{\Omega}_j), j \geq 0$ . From Lemma 2.4,



$V_0$  is contained in the Fatou set  $F(f_0) = \overline{\mathbb{C}} \setminus J(f_0)$  and  $f_0 : \Omega_{j+1} \rightarrow \Omega_j$  is a homeomorphism for  $j \geq 0$ .

**2.4. Definition of quasiconformality.** In [KK], S. Kallunki and P. Koskela improved the remarkable result about the definition of quasiconformal map in Euclidean space in [HK]. We state it in Riemman sphere case with the original inequality replaced by an equivalent condition. This is because that the original inequality appeared in [KK] holds when  $r \rightarrow 0$ . Then we may use this inequality in spherical metric. Moreover, we can replace the basis of disks and its image by that of uniformly bounded distortion.

By definition, a subset  $E \subset \overline{\mathbb{C}}$  of  $\sigma$ -finite length is the countable union of subsets of finite one dimensional Hausdorff measure in spherical metric (Cf. [G]).

**Lemma 2.5.** *Let  $\phi : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  is a homeomorphism. If there are a set  $E$  of  $\sigma$ -finite length and a constant  $K$  such that there exists a basis of neighborhoods  $\{U(x)\}$  of  $x$  with  $\text{Dist}(U(x), x) < K$  and  $\text{Dist}(\phi(U(x)), \phi(x)) < K$  for each  $x \in \overline{\mathbb{C}} \setminus E$ , then  $\phi$  is quasiconformal.*

### 3. PROOFS OF MAIN RESULTS

**3.1. Proof of Theorem 1.1.** Suppose that  $J(f) \neq \overline{\mathbb{C}}$ . One can find a spherical disk in  $\overline{\mathbb{C}}$  of radius  $\varepsilon_0$  disjoint from  $J(f)$ .

For  $x \in J(f)$ ,  $r > 0$ , define

$$h(x, r) = \sup\{\xi \mid \text{there is } B(y, \xi) \subset F(f) \cap B(x, r)\}.$$

Then  $h : J(f) \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is continuous.

Since  $\text{int}(J(f)) = \emptyset$ ,

$$h(r) = \inf\{h(x, r) \mid x \in J(f)\} > 0$$

for any  $r > 0$ .

First of all, we suppose  $f$  is a critically non-recurrent rational map having no parabolic periodic points. From Lemma 2.3, there are  $0 <$

## APPLICATIONS OF MAÑÉ'S THEOREM

$\delta_0 < \frac{\pi - \varepsilon_0}{2}$  and  $p < \infty$  so that for any  $x \in J(f)$ ,  $n > 0$  and any component  $V$  of  $f^{-n}(B(x, 2\delta_0))$ ,  $V$  is simply connected domain missing a spherical disk of radius of  $\varepsilon_0$ , and

$$\deg(f^n : V \rightarrow B(x, 2\delta_0)) \leq p.$$

For any  $z_0 \in J(f)$  and  $0 < r \leq \delta_0$ , look at the forward images  $B_m = f^m(B(z_0, r))$ ,  $z_m = f^m(z_0)$ ,  $m \geq 0$ .

Let  $n_0$  be the smallest integer so that  $\text{diam} B_{n_0+1} > \delta_0$ . Then  $l_0 \delta_0 \leq \text{diam} B_{n_0} \leq \delta_0$  for some  $0 < l_0 < 1$ . By Corollary 2.1, there are a constant  $K$  depending only on  $p$  and  $\varepsilon$ , and a disk  $B(z_{n_0}, r_0) \subset B_{n_0}$  with  $r_0 \geq \text{diam} B_{n_0}/K \geq l_0 \delta_0$  such that the component of  $f^{-n_0}(B(z_{n_0}, r_0))$  containing  $z_0$  is a subset of  $B(z_0, r)$ . There exists  $B(y_0, \frac{1}{2}h(r_0)) \subset B(z_{n_0}, r_0) \cap F(f)$ . Let  $D_0$  be a component of  $f^{-n_0}(B(y_0, \frac{1}{2}h(r_0)))$  contained in  $B(z_0, r)$ ,  $y \in D_0 \cap f^{-n_0}(y_0)$ . Then  $\text{Dist}(\partial D_0, y) \leq M$  for some  $M$  depending only on  $p$  and  $\text{diam} D_0 \asymp r$ . Therefore, there exists  $0 < k' < 1$  which does not depend on  $z_0$  and  $r$  so that  $B(y, k'r) \subset D_0 \subset B(z_0, r) \cap F(f)$ .

Now let  $f$  be a non-recurrent rational map having parabolic periodic points. For simplicity, we suppose  $f$  has only one non-degenerate parabolic fixed point 0.

Take  $\Omega_j (j \geq 0)$  as in Section 2.3. The set  $X_0 = J(f) \setminus (\cup_{j \geq 1} \Omega_j \cup \{0\})$  is a compact subset of  $J(f)$ . By Lemma 2.3, there exist  $\delta_1 > 0$  and  $p_1 < \infty$  so that for all  $x \in X_0$ ,  $n > 0$ ,  $0 < \delta \leq \delta_1$  and any component  $V$  of  $f^{-n}(B(x, 2\delta))$ ,  $V$  is simply connected and

$$\deg(f^n : V \rightarrow B(x, 2\delta)) \leq p_1.$$

Let  $K_1$  be the constant depending only on  $p_1$  and  $\varepsilon_0$  in Corollary 2.1. Suppose both  $\text{diam} D$  and  $\text{diam}(f(D))$  are less than  $\delta$ . If  $D$  contains a disk  $B(z, \epsilon)$  with  $\epsilon \geq \text{diam} D/K_1$ , then  $f(D)$  contains a disk  $B(f(z), \epsilon^*)$  such that the component of  $f^{-1}(B(f(z), \epsilon^*))$  containing  $z$  is a subset of  $B(z, \epsilon)$ , where  $\epsilon^* \geq \text{diam} f(D)/K_0$  for a constant  $K_0$  depending only on  $K_1$  and  $\varepsilon_0$ .

YESHUN SUN AND YONGCHENG YIN

Choose  $N$  large enough so that  $N \geq 128K_0R_0$  and  $d(0, \partial\Omega_n) > 4K_0\text{diam}\Omega_n$  for  $n \geq N$ . Take  $\delta_2 = \eta\text{diam}\Omega_N$  for a suitable  $\eta$  so that

$$B(x, 2\delta_2) \subset V_0^* = V_0 \cup (\cup_{j \geq 0} \Omega_j)$$

for any  $x \in \cup_{j=1}^N \Omega_j$ .

Now we prove that the Julia set  $J(f)$  is shallow.

For any  $z_0 \in J(f)$  and  $0 < r \leq \delta_0 = \min(\delta_1, \delta_2)$ , denote  $B_m = f^m(B(z_0, r))$ ,  $z_m = f^m(z_0)$ ,  $m \geq 0$ . Let  $n_0$  be the smallest integer such that  $\text{diam}B_{n_0+1} > \delta_0$ . Then  $\text{diam}B_m \leq \delta_0$  for  $m \leq n_0$  and  $\text{diam}B_{n_0} \asymp \delta_0$ .

If  $z_{n_0} \in X_0$ , then the same argument as in no parabolic periodic points case shows there exists  $B(y, k_1r) \subset B(z_0, r) \cap F(f)$  for some  $0 < k_1 < 1$ .

Suppose  $z_{n_0} \in (\cup_{j \geq 1} \Omega_j) \cup \{0\}$  and  $m_0 \geq 0$  is the smallest integer such that  $z_j \in (\cup_{j \geq 1} \Omega_j) \cup \{0\}$  for all  $m_0 \leq j \leq n_0$ . Then  $B_{m_0}$  contains a disk  $W = B(z_{m_0}, \epsilon^*)$  with  $\epsilon^* \geq \text{diam}B_{m_0}/K_0$  such that the component of  $f^{-m_0}(W)$  containing  $z_0$  is a subset of  $B(z_0, r)$ .

We claim that there exists a disk  $B(y_0, r_0) \subset B_{m_0} \cap F(f)$  such that  $f^{-m_0}(B(y_0, r_0))$  has a component contained in  $B(z_0, r)$  for some  $r_0 \asymp \text{diam}B_{m_0}$ .

*Case 1.* If  $z_{m_0} = 0$ , take  $r_0 = \frac{1}{2}\epsilon^*$ .

*Case 2.* If  $z_{n_0} \in \Omega_{j_0}$  for  $1 \leq j_0 \leq N$ , then  $B_{n_0} \subset B(z_{n_0}, \delta_0)$  and  $B(z_{n_0}, 2\delta_0) \subset V_0^*$ . Let  $B'$  be a component of  $f^{-(n_0-m_0)}(B(z_{n_0}, 2\delta_0))$  containing  $B_{m_0}$ . Then  $f^{n_0-m_0} : B' \rightarrow B(z_{n_0}, 2\delta_0)$  is univalent and  $\text{mod}(B' \setminus B_{m_0}) \geq \frac{1}{2\pi} \log 2$ . By distortion theorem of univalent map,  $\text{Dist}(\partial W_1, z_{n_0}) \asymp 1$  and there is  $B(y_0^*, r_0^*) \subset W_1 \cap F(f)$  for some  $r_0^* \asymp \delta_0$ , where  $W_1 = f^{n_0-m_0}(W)$ . Hence there is  $B(y_0, r_0) \subset W \cap F(f)$  for some  $r_0 \asymp \text{diam}B_{m_0}$ .

*Case 3.* If  $z_{n_0} \in \Omega_{j_0}$  for  $j_0 > N$ ,  $z_{m_0} \in \Omega_{j_0} + (n_0 - m_0) = \Omega_{i_0}$ .

(3a). When  $\text{diam}B_{m_0} > 2K_0\text{diam}\Omega_{i_0}$ ,  $d(z_{m_0}, \partial W) > 2\text{diam}\Omega_{i_0}$ , take  $r_0 = \frac{1}{4}d(z_{m_0}, \partial W)$ .

## APPLICATIONS OF MANÉ'S THEOREM

(3b). When  $\text{diam } B_{m_0} \leq 2K_0 \text{diam } \Omega_{i_0}$ ,  $D = D(z_{m_0}, 2\text{diam } B_{m_0}) \subset V^*$ . Let  $\tilde{D} = \Phi_-(D)$ ,  $\tilde{W} = \Phi_-(W)$ ,  $\tilde{W}_1 = \Phi_-(W_1)$  and  $\tilde{z}_{m_0} = \Phi_-(z_{m_0})$ . Then  $\text{Dist}_E(\partial\tilde{W}, \tilde{z}_{m_0}) \leq 16 \text{Dist}(\partial W, z_{m_0}) \leq 16K_0$ .

If  $d_E(\tilde{z}_{m_0}, \partial\tilde{W}) > 4R_0$ , then there is a disk  $\tilde{B} = D(\tilde{y}_0, \frac{1}{2}d_E(\tilde{z}_{m_0}, \partial\tilde{W})) \subset \tilde{V}_0$  such that  $\Phi_-^{-1}(\tilde{B}) = B$  contains a disk  $B(y_0, r_0) \subset W \cap V_0 \subset W \cap F(f)$  for  $r_0 \asymp \text{diam } B_{m_0}$ .

If  $d_E(\tilde{z}_{m_0}, \partial\tilde{W}) \leq 4R_0$ , then  $\tilde{W} \subset D(\tilde{z}_{m_0}, 64K_0R_0)$  and  $\tilde{W}_1 = \tilde{W} + (n_0 - m_0) \subset D(\tilde{z}_{n_0}, 64K_0R_0)$ ,  $\tilde{z}_{n_0} = \tilde{z}_{m_0} + (n_0 - m_0) = \Phi_-(z_{n_0})$ . Since  $\tilde{z}_{n_0} \in \tilde{\Omega}_{j_0}$ ,  $j_0 > N \geq 128K_0R_0$ , we have  $\tilde{D}' = D(\tilde{z}_{n_0}, 128K_0R_0) \subset \tilde{V}_0^*$ ,  $D' = \Phi_-^{-1}(\tilde{D}') \subset V^*$ ,  $B_{n_0} \subset D'$  and  $\text{mod}(D' \setminus B_{n_0}) \geq \frac{1}{2\pi} \log 2$ . Hence  $\text{Dist}(\partial W_1, z_{n_0}) \leq 16 \text{Dist}_E(\partial\tilde{W}_1, \tilde{z}_{n_0}) = 16 \text{Dist}_E(\partial\tilde{W}, \tilde{z}_{m_0}) \leq 16^2 K_0$ .

Let  $D''$  be the component of  $f^{-(n_0-m_0)}(D')$  containing  $B_{m_0}$ . Then  $f^{n_0-m_0} : D'' \rightarrow D'$  is univalent. By distortion theorem, there is a disk  $B(y_0, r_0) \subset W \cap F(f)$  for some  $r_0 \asymp \text{diam } B_{m_0}$ .

The proof of our claim is completed.

Now we come back to the proof of the main theorem.

If  $m_0 = 0$ , we are done. If  $m_0 \geq 1$ , then  $B_{m_0-1} \cap f^{-1}(B(y_0, r_0))$  contains a disk with radius  $r'_0 \asymp \text{diam } B_{m_0-1}$ . The same argument as in no parabolic points case shows that there is  $B(y, k_2 r) \subset B(z_0, r) \cap F(f)$  for some  $0 < k_2 < 1$ . Take  $k' = \min(k_1, k_2)$ . Then for any  $z_0 \in J(f)$ ,  $r > 0$ ,  $B(z_0, r)$  contains a disk  $B(y, k' r) \subset F(f)$  and hence  $J(f)$  is shallow.

**3.2. Proof of Theorem 1.2.** Suppose that  $f$  and  $g$  are two critically non-recurrent rational maps which are conjugated by a homeomorphism  $\psi$ . By a routine argument on Fatou set,  $\psi$  is isotopic relative to  $J(f)$  to a new conjugacy  $\phi$  which is quasiconformal on Fatou set  $F(f)$ . We shall prove that  $\phi$  is actually a globally quasiconformal map.

Let  $X_0(f) \subset J(f)$  (resp.  $X_0(g) \subset J(g)$ ) be the compact sets defined in the proof of Theorem 1.1. Let  $\delta_f(\delta_g, \text{resp.}) < \frac{\pi}{2}$  (We will apply

Corollary 2.1 for  $\varepsilon_0 = \frac{\pi}{2}$ ) and integers  $p(f)(p(g)$ , resp.) be the numbers guaranteed by Mañé's theorem for  $X_0(f)(X_0(g)$ , resp.) and for  $\varepsilon = \frac{\pi}{2}$ . In cases  $J(f) = \overline{\mathbb{C}}$  and  $J(f)$  containing no parabolic points, let  $X_0(f) = J(f)$ .

Take a small  $0 < r_0 < \frac{1}{2}\delta_f$  such that for all  $x \in J(f)$ ,  $\phi(B(x, r_0)) \subset B(\phi(x), \frac{1}{2}\delta_g)$ .

Let  $x \in J(f)$  and  $U_n(x)$  be a component of  $f^{-n}(B(f^n(x), r_0))$  containing  $x$ . There is a constant  $K_1$  does not depend on  $y$  such that  $\text{Dist}(\phi(B(y, r_0)), \phi(y)) \leq K_1$ . If  $x$  is not eventually mapped to the parabolic periodic points, it visits  $X_0(f)$  infinitely many times. It follows from the distortion theorem for  $p$ -valent version that there are subsequence  $\{n_k\}$  and a constant  $K$  independent of  $x$  such that

$$\text{Dist}(U_{n_k}(x), x) \leq K$$

and

$$\text{Dist}(\phi(U_{n_k}(x)), \phi(x)) \leq K.$$

It then follows from Lemma 2.5 that  $\phi : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$  is quasiconformal. The proof is complete.

**3.3. Proof of Theorem 1.3.** By Corollary 1.1, we assume the Julia set of  $f$  is the whole sphere.

Let  $X \subset \overline{\mathbb{C}}$  be a completely invariant subset. Suppose that  $\text{mes}(X) > 0$ . We shall prove that  $X$  has the full measure.

Let  $x_0 \in X$  be a density point of  $X$ . Let  $x_n = f^n(x)$ ,  $n > 0$ . We can choose a subsequence  $\{x_{n_k}\}$  converging to  $x^*$ . Let  $A = \{x_{n_k}\} \cup \{x^*\}$ . Then  $A$  is a proper compact subset of  $J(f)$ . Let  $\delta$  and  $p$  be the numbers given for  $\varepsilon$  in Mañé's theorem with  $\varepsilon$ -neighborhood of  $A$  missing a spherical disk of radius  $\varepsilon_0$  and  $U_{n_k}$  be the component of  $f^{-n_k}(B(x_{n_k}, \frac{1}{2}\delta))$  containing  $x_0$ . Then  $U_{n_k}$  shrinks to  $x_0$  nicely since  $\text{Dist}(U_{n_k}, x_0) \leq K$ ,  $K$  depending only on  $p$  and  $\varepsilon_0$ . Therefore,

$$\lim_{k \rightarrow \infty} \frac{\text{mes}(U_{n_k} \cap X)}{\text{mes}(U_{n_k})} = 1,$$

## APPLICATIONS OF MAÑÉ'S THEOREM

which is equivalent to

$$\lim_{k \rightarrow \infty} \frac{\text{mes}(U_{n_k} \cap (\overline{\mathbb{C}} \setminus X))}{\text{mes}(U_{n_k})} = 0.$$

It follows from distortion theorem that,

$$\lim_{k \rightarrow \infty} \frac{\text{mes}(B(x_{n_k}, \frac{1}{2}\delta) \cap (\overline{\mathbb{C}} \setminus X))}{\text{mes}(B(x_{n_k}, \frac{1}{2}\delta))} = 0,$$

which is equivalent to

$$\lim_{k \rightarrow \infty} \frac{\text{mes}(B(x_{n_k}, \frac{1}{2}\delta) \cap X)}{\text{mes}(B(x_{n_k}, \frac{1}{2}\delta))} = 1,$$

and hence

$$\frac{\text{mes}(B(x^*, \frac{1}{2}\delta) \cap X)}{\text{mes}(B(x^*, \frac{1}{2}\delta))} = 1.$$

This implies that  $B(x^*, \frac{1}{2}\delta) \subset X$  a.e.

Since  $B(x^*, \frac{1}{2}\delta) \subset \overline{\mathbb{C}} = J(f)$ , there exists an integer  $k$  such that  $f^k(B(x^*, \frac{1}{2}\delta)) = \overline{\mathbb{C}}$ . Hence  $\overline{\mathbb{C}} \subset X$  a.e. This means that  $X$  has full measure and completes the proof.

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YESHUN SUN AND YONGCHENG YIN

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